OCR Core 1

Coordinate Geometry

Section 2: Circles

Notes and Examples

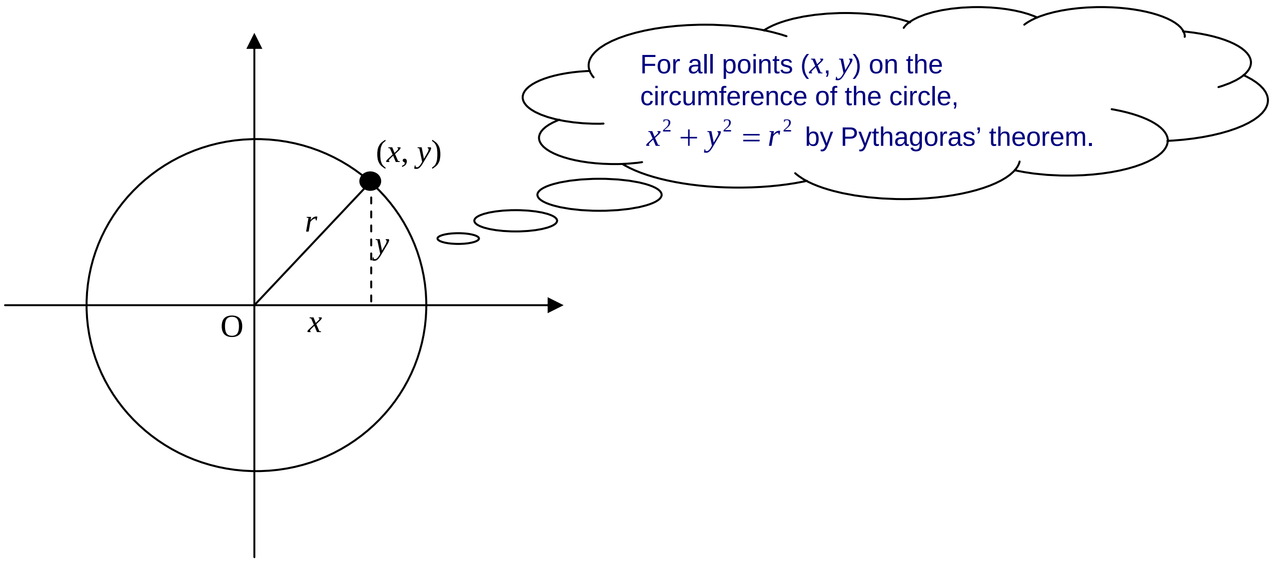
These notes and examples contain subsections on

- **The equation of a circle**
- **Finding the equation of a circle**
- **Circle geometry**
- **The intersection of a line and a circle**

The equation of a circle

Start this section by looking at the **Circles dynamic spreadsheet**. Select the *Circle Equations* sheet. First, set the centre of the circle to be the origin and vary the radius. Look at how the equation of the circle changes.

A circle with centre the origin and radius r is the locus of all points whose distance from the origin is r units.



The general equation of a circle, centre (0, 0) and radius r is $x^2 + y^2 = r^2$.

Now use the Circles dynamic spreadsheet again. This time vary the coordinates of the centre of the circle, and look at how the equation of the circle changes.

A circle with centre (a, b) and radius r is the locus of all points whose distance from the point (a, b) is r units.

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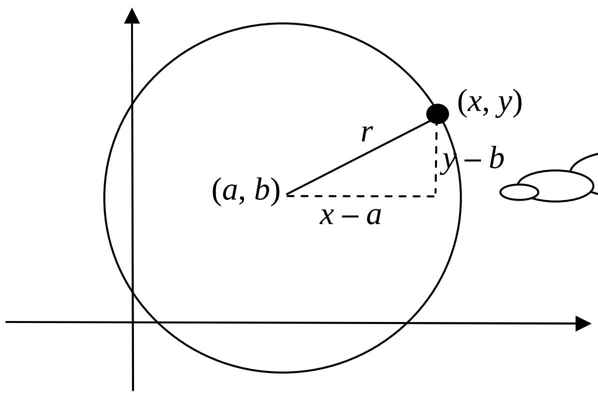


For all points (x, y) on the circumference of the circle, $(x-a)^2+(y-b)^2=r^2$ by Pythagoras' theorem.

The general equation of a circle, centre (a, b) and radius r is $(x - a)^2 + (y - b)^2 = r^2$.

You can also explore equations of circles using the Flash resources **Equation of a circle centre O** and **Equation of a circle centre (a, b)**.

Example 1

For each of the following circles find (i) the coordinates of the centre and (ii) the radius.

(a) $x^2 + y^2 = 49$

(b) $(x + 2)^2 + (y - 6)^2 = 9$

This is a particular case of the general form $x^2 + y^2 = r^2$ which has centre (0, 0) and radius r.

Solution

(a) $x^2 + y^2 = 49$ can be written as $x^2 + y^2 = 7^2$.

(i) The coordinates of the centre are (0, 0)

(ii) The radius is 7.

(b) $(x + 2)^2 + (y - 6)^2 = 9$ can be written as $(x - (-2))^2 + (y - 6)^2 = 3^2$.

(i) The coordinates of the centre are (–2, 6)

(ii) The radius is 3.

This is a particular case of the general form $(x - a)^2 + (y - b)^2 = r^2$ which has centre (a, b) and radius r.

For practice in examples like the one above, try the interactive questions **Finding the radius and centre of a circle** (*circle equation in its simplest form*).

Sometimes the circle equation needs to be rearranged into its standard form before you can find the centre and radius. To do this, you need to complete the square on the x terms and on the y terms (completing the square is covered in chapter 4).

Example 2

Show that the equation $x^2+y^2+4x-6y-3=0$ represents a circle, and find its centre and radius.

Solution

$x^2 + y^2 + 4x - 6y - 3 = 0$

$x^2 + 4y + y^2 - 6y = 3$

$(x+2)^2 - 4 + (y-3)^2 - 9 = 3$

$(x+2)^2 + (y-3)^2 = 16$

This is the equation of a circle, centre (-2, 3), radius 4.

Alternatively, you can use the approach in the book involving the general equation of the circle (see Example 13.1.6 on page 199).

For practice in examples like the one above, try the interactive resource **Finding the radius and centre of a circle** (*circle equation in its expanded form*).

Finding the equation of a circle

In Section 1 you looked at different ways of finding the equation of a line. You can find the equation of a line from the gradient and the intercept, or from the gradient and one point on the line, or from two points on the line.

In the same way, there are several ways of finding the equation of a circle, depending on the information available.

Finding the equation of a circle from the radius and centre

Example 3

Find the equation of each of the following.

(a) a circle, centre (0, 0) and radius 4.

(b) a circle, centre (3, –4) and radius 6.

Solution

(a) The equation of a circle centre the origin is $x^2 + y^2 = r^2$

$r = 4$ so the equation is $x^2 + y^2 = 4^2$

i.e. $x^2 + y^2 = 16$

(b) The equation of a circle centre (a, b) and radius r is $(x - a)^2 + (y - b)^2 = r^2$

$a = 3$, $b = -4$ and $r = 6$ so the equation is $(x - 3)^2 + (y - (-4))^2 = 6^2$

i.e. $(x - 3)^2 + (y + 4)^2 = 36$

Finding the equation of a circle from its centre and one point on its circumference

If you know the centre of the circle and one point on its circumference, you can find the radius by calculating the distance between these two points. You can then find the equation of the circle.

Example 4
Find the equation of the circle, centre (1, -2), which passes through the point (-2, -3).

Solution
The distance *r* between (1, -2) and (-2, -3) is given by:

$$r = \sqrt{(1-(-2))^2 + (-2-(-3))^2}$$
$$= \sqrt{3^2 + 1^2}$$
$$= \sqrt{10}$$

The radius of the circle is therefore $\sqrt{10}$.
The equation of the circle is $(x-1)^2 + (y+2)^2 = 10$

For practice in examples like the one above, try the interactive resource **Find the equation of a circle**.

Finding the equation of a circle from three points on its circumference

To find the equation of a line, you need the coordinates of two points on the line. To find the equation of a circle, you need the coordinates of three points on the circumference of the circle.

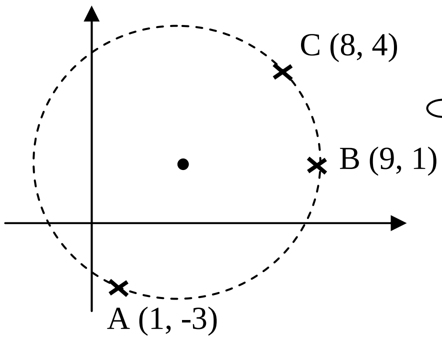
One method is illustrated by the **Circles dynamic spreadsheet**. Select the sheet *Circumcentre* and follow the instructions on the sheet. This demonstration shows that the centre of the circle is the intersection of the perpendicular bisector of each pair of points.

To find the centre of a circle through three points A, B and C, it is sufficient to find two of the perpendicular bisectors. For example, you can find the equations of the perpendicular bisectors of AB and BC, and then solve these equations simultaneously to find the point of intersection, i.e. the centre of the circle.

You can then use the coordinates of the centre and one of the three points A, B and C to find the radius of the circle (as in Example 4), and hence find the equation of the circle.

Example 5
Find the equation of the circle passing through A (1, –3), B (9, 1) and C (8, 4).

Solution



A sketch is often helpful. The sketch does not need to be accurate. It gives some idea of roughly where the centre is, so you can check your answer is reasonable.

You want to find the equation of the perpendicular bisector of AB. This is perpendicular to AB and passes through the midpoint M of AB.

The gradient of AB is found by using $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

$$m = \frac{1-(-3)}{9-1} = \frac{4}{8} = \frac{1}{2}$$

Note: Looking at the sketch we expect the gradient of AB to be positive.

Using $m_1 m_2 = -1$, the gradient of the perpendicular bisector is -2.

The midpoint M of AB is found by using M = $\left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right)$.

You are given A(1, -3) and B(9, 1) so M is $\left(\dfrac{1+9}{2}, \dfrac{-3+1}{2}\right) = (5, -1)$

The perpendicular bisector is found using $y - y_1 = m(x - x_1)$ with $(x_1, y_1) = (5, 1)$ and $m = -2$.

so $y - (-1) = -2(x - 5)$

$y + 1 = -2x + 10$

$y = -2x + 9$ (equation I)

Next, use the same method to find the perpendicular bisector of BC.

The gradient of BC is $\dfrac{4-1}{8-9} = -3$

Note: Looking at the sketch, we expect the gradient of BC to be negative.

Therefore the gradient of the perpendicular bisector of BC is $\dfrac{1}{3}$.

The midpoint N of BC is $\left(\frac{9+8}{2}, \frac{1+4}{2}\right)$ so N is (8.5, 2.5).

The equation of the perpendicular bisector is $y - 2.5 = \frac{1}{3}(x - 8.5)$

$$3(y - 2.5) = x - 8.5$$
$$3y - 7.5 = x - 8.5$$
$$3y = x + 1 \quad \text{(equation II)}$$

$y = -2x + 9$ (equation I)
$3y = x + 1$ (equation II)

Next, find the coordinates of the centre of the circle by solving equations (I) and (II) simultaneously.

Substituting (I) into (II)
$$3(-2x + 9) = x + 1$$
$$-6x + 27 = x + 1$$
$$28 = 7x$$
$$x = 4$$

Substituting $x = 4$ into equation (I) gives $y = -2(4) + 9 = 1$
So the coordinates of the centre are (4, 1).

Note: Looking at the sketch this appears to be a plausible result.

The radius is the distance between the centre (4, 1) and a point on the circumference such as (9, 1). This can be found by using $d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$.

$$\text{radius} = \sqrt{(9-4)^2 + (1-1)^2} = \sqrt{25} = 5$$

Finally, using the general form $(x - a)^2 + (y - b)^2 = r^2$ with $a = 4$, $b = 1$ and $r = 5$ the equation of the circle is

$$(x - 4)^2 + (y - 1)^2 = 25.$$

Note: You should check that each of the points A, B and C satisfy this equation.

Circle geometry

There are three important facts about circles that you need to know. These facts often help to solve problems involving circles.

1. The angle in a semicircle is a right angle.
   See the Flash resource **The angle in a semicircle** for a demonstration.

2. The perpendicular from the centre of a circle to a chord bisects the chord.
   See the Flash resource **Perpendicular to a chord** for a demonstration.

3. The tangent to a circle is perpendicular to the radius at that point
See the Flash resource **Tangent and radius** for a demonstration.

Keep these properties in mind when dealing with problems involving circles. Sometimes using these properties can make solving problems very easy!

As for the work in section 1, you should draw a sketch diagram when solving problems involving coordinate geometry.

The intersection of a line and a circle

Look at the **Circles dynamic spreadsheet**. Select the sheet *Circle and a line*. Try varying the equation of the line and/or the circle, and make sure that you can see that there may be two intersections, no intersections or one intersection (in which case the line touches the circle).

You can also look at the Flash resource **Intersection of a circle and a line**.

Example 6

Find the coordinates of the point(s) where the circle $(x+2)^2 + (y-1)^2 = 9$ meets

(i) the line $y = 5$

(ii) the line $x = 1$

(iii) the line $y = 2 - x$

Solution

(i) Substituting $y = 5$ into the equation of the circle:

$$(x+2)^2 + (5-1)^2 = 9$$
$$(x+2)^2 + 16 = 9$$
$$(x+2)^2 = -7$$

The expression $(x + 2)^2$ cannot be negative

There are no solutions. The line does not meet the circle.

(ii) Substituting $x = 1$ into the equation of the circle:

$$(1+2)^2 + (y-1)^2 = 9$$
$$9 + (y-1)^2 = 9$$
$$(y-1)^2 = 0$$
$$y = 1$$

The point is on the line $x = 1$, so its x-coordinate must be 1.

The line touches the circle at (1, 1).

(iii) Substituting $y = 2 - x$ into the equation of the circle:

$$(x+2)^2 + (2-x-1)^2 = 9$$
$$(x+2)^2 + (1-x)^2 = 9$$
$$x^2 + 4x + 4 + 1 - 2x + x^2 = 9$$
$$2x^2 + 2x - 4 = 0$$

$$x^2 + x - 2 = 0$$
$$(x-1)(x+2) = 0$$
$$x = 1 \text{ or } x = -2$$

When $x = 1$, $y = 2 - 1 = 1$
When $x = -2$, $y = 2 - (-2) = 4$

Substitute the x values into the equation of the line to find the y-coordinates.

The line crosses the circle at (1, 1) and (–2, 4).

For practice in examples like the one above, try the interactive questions **Circle and line intersection**.