# OCR Further Pure 1

## Complex Numbers

## Section 1: Introduction to complex numbers

### Notes and Examples

These notes contain subsections on

- The number system
- Adding and subtracting complex numbers
- Multiplying complex numbers
- Complex conjugates
- Equations with complex roots
- Equating real and imaginary parts

### The number system

In your learning of mathematics, you have come across different types of number at different stages. Each time you were introduced to a new set of numbers, this allowed you to solve a wider range of problems.

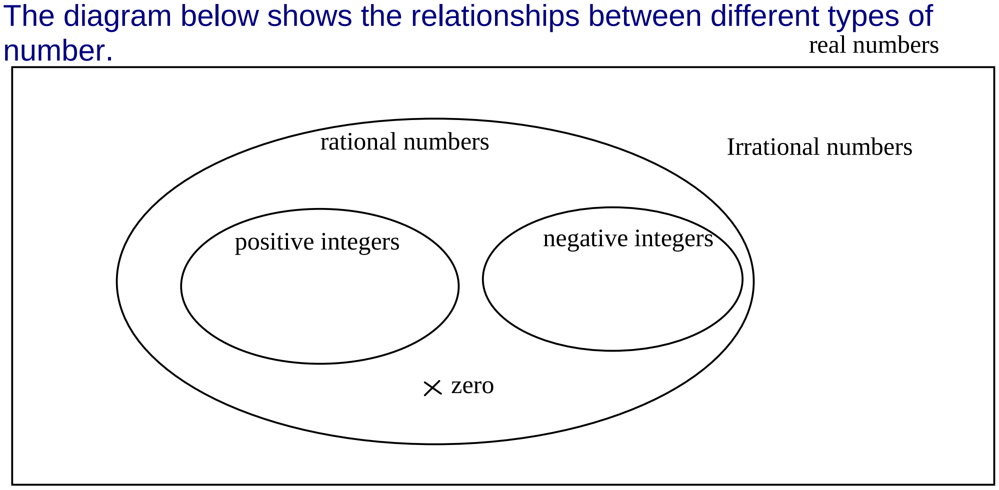
The first numbers that you came across were the counting numbers (natural numbers). These allowed you to solve equations like $x + 2 = 5$.

Later you would meet negative numbers, which allowed you to solve equations like $x + 5 = 2$, and rational numbers, which meant you could solve equations like $2x = 5$.

When irrational numbers were included, you could solve equations like $x^2 = 2$.

However, there are still equations which you cannot solve, such as $x^2 = -4$. You know that there are no real numbers which satisfy this equation. However, this equation, and others like it, can be solved using imaginary numbers, which are based on the number i, which is defined as $\sqrt{-1}$.

The diagram below shows the relationships between different types of number.



**Example 3**

Write $\frac{2+3i}{3-4i}$ in the form $a + bi$.

**Solution**

$$\frac{2+3i}{3-4i} = \frac{(2+3i)(3+4i)}{(3-4i)(3+4i)}$$
$$= \frac{6+8i+9i-12}{9+16}$$
$$= \frac{-6+17i}{25}$$
$$= -\tfrac{6}{25} + \tfrac{17}{25}i$$

Multiply top and bottom by 3 + 4i (the complex conjugate of 3 – 4i)

For practice in examples like the one above, try the interactive resource **Dividing complex numbers**.

## Equations with complex roots

When you first learned to solve quadratic equations using the quadratic formula, you found that some quadratic equation had no real solutions. However, using complex numbers you can find solve all quadratic equations.

**Example 4**

Solve the quadratic equation

$$x^2 + 6x + 13 = 0$$

**Solution**

Using the quadratic formula with $a = 1$, $b = 6$, $c = 13$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-6 \pm \sqrt{36 - 4 \times 1 \times 13}}{2 \times 1}$$
$$= \frac{-6 \pm \sqrt{-16}}{2}$$
$$= \frac{-6 \pm 4i}{2}$$
$$= -3 \pm 2i$$

The solutions of the equation are $x = -3 + 2i$ and $x = -3 - 2i$

Notice that the quadratic equation in Example 4 has two complex solutions which are a pair of complex conjugates. All quadratic equations with real coefficients have two solutions: either two real solutions (which could be a repeated solution) or two complex solutions which are a pair of complex conjugates.

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The next example shows how you can find a quadratic equation with roots at particular complex values. A quadratic equation with roots at $x = a$ and $x = b$ can be written as $(x - a)(x - b) = 0$, and this also applies to situations where the roots are complex numbers.

**Example 5**
Find the quadratic equation which has roots at $x = 4 + 2\mathrm{i}$ and $x = 4 - 2\mathrm{i}$.

**Solution**

$$(x - (4 + 2\mathrm{i}))(x - (4 - 2\mathrm{i})) = 0$$
$$(x - 4 - 2\mathrm{i})(x - 4 + 2\mathrm{i}) = 0$$
$$(x - 4)^2 - (2\mathrm{i})^2 = 0$$
$$x^2 - 8x + 16 + 4 = 0$$
$$x^2 - 8x + 20 = 0$$

The two middle terms cancel out since this expression is of the form $(x - a)(x + a)$

The Flash resource **Working with complex numbers** tests you on multiplication, complex conjugates and equations with complex roots.

## Equating real and imaginary parts

For two complex numbers to be equal, then the real parts must be equal and the imaginary parts must be equal. So one equation involving complex numbers can be written as two equations, one for the real parts, one for the imaginary parts.

The example below shows how this technique can be used to solve equations involving complex numbers. Two solutions are shown. In the first solution, the equation is treated in the same sort of way as for an equation involving real numbers, so division of complex numbers is used. In the second solution, the method of equating real and imaginary parts is used.

**Example 6**
Solve the equation

$$(3 - 2\mathrm{i})(z - 1 + 4\mathrm{i}) = 7 + 4\mathrm{i}$$

**Solution 1**

$$(3 - 2\mathrm{i})(z - 1 + 4\mathrm{i}) = 7 + 4\mathrm{i}$$
$$z - 1 + 4\mathrm{i} = \frac{7 + 4\mathrm{i}}{3 - 2\mathrm{i}} = \frac{(7 + 4\mathrm{i})(3 + 2\mathrm{i})}{(3 - 2\mathrm{i})(3 + 2\mathrm{i})}$$

Divide both sides by $3 - 2\mathrm{i}$

$$= \frac{21 + 14\mathrm{i} + 12\mathrm{i} - 8}{9 + 4}$$
$$= \frac{13 + 26\mathrm{i}}{13} = 1 + 2\mathrm{i}$$
$$z = 1 + 2\mathrm{i} - (-1 + 4\mathrm{i})$$
$$= 2 - 2\mathrm{i}$$

Subtract $-1 + 4\mathrm{i}$ from each side

**Solution 2**

Let $z = x + \mathrm{i}y$

$$(3 - 2\mathrm{i})(x + \mathrm{i}y - 1 + 4\mathrm{i}) = 7 + 4\mathrm{i}$$
$$(3 - 2\mathrm{i})((x - 1) + \mathrm{i}(y + 4)) = 7 + 4\mathrm{i}$$
$$3(x - 1) - 2\mathrm{i}(x - 1) + 3\mathrm{i}(y + 4) - 2\mathrm{i}^2(y + 4) = 7 + 4\mathrm{i}$$
$$3(x - 1) - 2\mathrm{i}(x - 1) + 3\mathrm{i}(y + 4) + 2(y + 4) = 7 + 4\mathrm{i}$$

Equating real parts: $3(x - 1) + 2(y + 4) = 7 \quad \Rightarrow 3x + 2y = 2$ ①

Equating imaginary parts: $-2(x - 1) + 3(y + 4) = 4 \quad \Rightarrow -2x + 3y = -10$ ②

① × 2 $\quad 6x + 4y = 4$

② × 3 $\quad -6x + 9y = -30$

Adding: $\quad 13y = -26$

$y = -2$

$x = 2$

$z = 2 - 2\mathrm{i}$