Further Pure Mathematics 1

Complex Numbers Section 3

Supplementary Notes and Examples

The modulus-argument form of complex numbers

You are familiar with describing a point in the plane using Cartesian coordinates. However, this is not the only way of describing the location of a point. One alternative is to give its distance from a fixed point (usually the origin) and a direction (in this case the angle between the line connecting the point to the origin, and the positive real axis). In the context of complex numbers this is usually called the modulus-argument form; in other contexts it is known as using polar coordinates.

This is a common method of describing locations in real life: you might say that a town is "50 miles north-west of London", or when walking in open countryside your map might show you that you need to walk 2 miles on a bearing of 124°.

In mathematics there are some situations in which this method of describing points is more convenient than Cartesian coordinates. In this section you will look at complex numbers in the modulus-argument form. There is more about polar coordinates and about the modulus-argument form of complex numbers in FP2.

To write a complex number z in modulus-argument form, all you need to do is to find the modulus, r, and the argument, θ. Then write the complex number as $z = r(\cos\theta + \mathrm{j}\sin\theta)$. Finding the modulus is straightforward enough, but finding the argument involves a little more work. It involves using some knowledge of Trigonometry from C2, including radians, and angles greater than 90°. However, if you haven't covered this work yet, don't worry. Click **here** for the Trigonometry notes which give some help on these topics. (If you have already done the chapter on Matrices in FP1, the work you did on rotation matrices will have given you some confidence in dealing with angles greater than 90°, though you still may need some help with radians.)

To find the argument of the complex number $z = x + y\mathrm{j}$, you need to find the value of $\arctan\frac{y}{x}$. Activity 2.8 shows that using a calculator to find $\arctan\frac{y}{x}$ gives angles, in radians, between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. This gives the correct angle for complex numbers in the first and fourth quadrants (i.e. those with positive real parts), but not for complex numbers in the second and third quadrants (i.e. those with negative real parts). To find an argument in the second quadrant, you need to add π to the answer the calculator gives, and to find an argument in the third quadrant, you need to subtract π from the answer your calculator gives. The Trigonometry Notes give a fuller explanation of this.

Activity 2.9 will help you to see whether you have understood this correctly!

Activity 2.11 reminds you of the values of sin, cos and tan for the common angles $\frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}$ (i.e. 30°, 45°, 60°). When you come across these angles, you are expected to use the exact values.

The enrichment Activity 2.12 shows one use of the modulus-argument form. The full solution to this Activity is given below.

Solution

(i) (a)
$$\begin{aligned} wz &= (1+\mathrm{j})(1-\sqrt{3}\mathrm{j}) \\ &= 1-\sqrt{3}\mathrm{j}+\mathrm{j}+\sqrt{3} \\ &= 1+\sqrt{3}+(1-\sqrt{3})\mathrm{j} \end{aligned}$$

(b)
$$\begin{aligned} \frac{w}{z} &= \frac{1+\mathrm{j}}{1-\sqrt{3}\mathrm{j}} \\ &= \frac{(1+\mathrm{j})(1+\sqrt{3}\mathrm{j})}{(1-\sqrt{3}\mathrm{j})(1+\sqrt{3}\mathrm{j})} \\ &= \frac{1+\sqrt{3}\mathrm{j}+\mathrm{j}-\sqrt{3}}{1+3} \\ &= \tfrac{1}{4}\left(1-\sqrt{3}+(1+\sqrt{3})\mathrm{j}\right) \end{aligned}$$

(ii) (a)
$$|w| = \sqrt{1^2+1^2} = \sqrt{2}$$
$$\arg w = \arctan 1 = \frac{\pi}{4}$$ ← w is in the first quadrant

(b)
$$|z| = \sqrt{1^2+3} = 2$$
$$\arg z = \arctan\left(-\sqrt{3}\right) = -\frac{\pi}{3}$$ ← z is in the fourth quadrant

(c)
$$|wz| = \sqrt{(1+\sqrt{3})^2+(1-\sqrt{3})^2} = \sqrt{1+2\sqrt{3}+3+1-2\sqrt{3}+3} = \sqrt{8} = 2\sqrt{2}$$
$$\arg(wz) = \arctan\left(\frac{1-\sqrt{3}}{1+\sqrt{3}}\right) = -\frac{\pi}{12}$$ ← wz is in the fourth quadrant

(d)
$$\left|\frac{w}{z}\right| = \tfrac{1}{4}\sqrt{(1-\sqrt{3})^2+(1+\sqrt{3})^2} = \tfrac{1}{4}\sqrt{8} = \frac{\sqrt{2}}{2}$$
$$\arg\left(\frac{w}{z}\right) = \arctan\left(\frac{1+\sqrt{3}}{1-\sqrt{3}}\right) = \frac{7\pi}{12}$$ ← $\frac{w}{z}$ is in the fourth quadrant

(iii)
$$|wz| = |w||z|, \left|\frac{w}{z}\right| = \frac{|w|}{|z|}$$
$$\arg(wz) = \arg w + \arg z, \arg\left(\frac{w}{z}\right) = \arg w - \arg z$$

(iv)
$$wz = r_1 r_2(\cos\theta_1 + \mathrm{j}\sin\theta_1)(\cos\theta_2 + \mathrm{j}\sin\theta_2)$$
$$= r_1 r_2(\cos\theta_1\cos\theta_2 + \mathrm{j}\cos\theta_1\sin\theta_2 + \mathrm{j}\sin\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2)$$
$$= r_1 r_2\left((\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + \mathrm{j}(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)\right)$$
$$= r_1 r_2\left(\cos(\theta_1+\theta_2) + \mathrm{j}\sin(\theta_1+\theta_2)\right)$$

This step involves using the compound angle formulae for sine and cosine:
$\sin(A \pm B) = \sin A\cos B \mp \sin B\cos A$
and
$\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$
If you have done the matrices chapter, you will have derived these in activity 1.10. You will also meet them in C4.

$$\frac{w}{z} = \frac{r_1(\cos\theta_1 + j\sin\theta_1)}{r_2(\cos\theta_2 + j\sin\theta_2)}$$
$$= \frac{r_1(\cos\theta_1 + j\sin\theta_1)(\cos\theta_2 - j\sin\theta_2)}{r_2(\cos\theta_2 + j\sin\theta_2)(\cos\theta_2 - j\sin\theta_2)}$$
$$= \frac{r_1(\cos\theta_1\cos\theta_2 - j\cos\theta_1\sin\theta_2 + j\sin\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2)}{(\cos^2\theta_2 + \sin^2\theta_2)}$$

$\cos^2\theta + \sin^2\theta = 1$

$$= \frac{r_1}{r_2}\left((\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) + \mathrm{j}(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2)\right)$$
$$= \frac{r_1}{r_2}\left(\cos(\theta_1-\theta_2) + \mathrm{j}\sin(\theta_1-\theta_2)\right)$$
$$|wz| = r_1 r_2, \left|\frac{w}{z}\right| = \frac{r_1}{r_2}, \arg(wz) = \theta_1 + \theta_2, \arg\left(\frac{w}{z}\right) = \theta_1 - \theta_2$$

This result gives a quick and easy way of multiplying and dividing complex numbers in the modulus-argument form. To multiply complex numbers, you multiply the moduli and add the arguments, and to divide the complex numbers you divide the moduli and subtract the arguments.

This topic is covered more fully in FP2.

Sets of points using the modulus-argument form

In the previous section, you looked at sets of points defined using the modulus of a complex number. You will now look at sets of points defined using the argument of a complex number.

All sets of points of the form $\arg(z-(a+b\mathrm{j}))=\theta$ consist of a half-line from the point $a + b\mathrm{j}$ in the direction θ (see Example 2.8 (i) and (ii)).

Sets of points of the form $\theta_1 \le \arg(z-(a+b\mathrm{j})) \le \theta_2$ consist of the two half-lines $\arg(z-(a+b\mathrm{j}))=\theta_1$ and $\arg(z-(a+b\mathrm{j}))=\theta_2$ and the region between them (see Example 2.8 (iii)).

Be careful with sets of points of the form $\arg(z-(a+b\mathrm{j})) \le \theta$ or $\arg(z-(a+b\mathrm{j})) \ge \theta$. Make sure that you know where the region starts and ends. Remember that the possible values of $\arg z$ are given by $-\pi < \arg z \le \pi$, and that any line which is not included in the set of points should be shown as dotted (see Example 1 below).

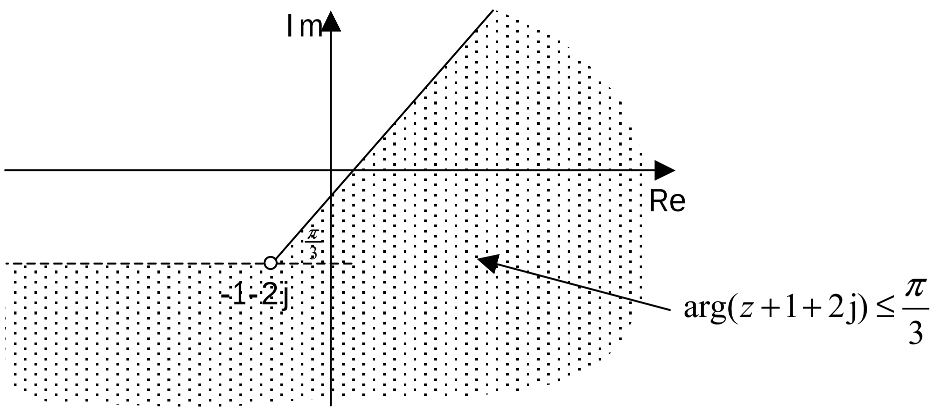
Example 1
Draw Argand diagrams to show each of the following sets of points.

(i) $\arg(z+1+2\mathrm{j}) \le \frac{\pi}{3}$

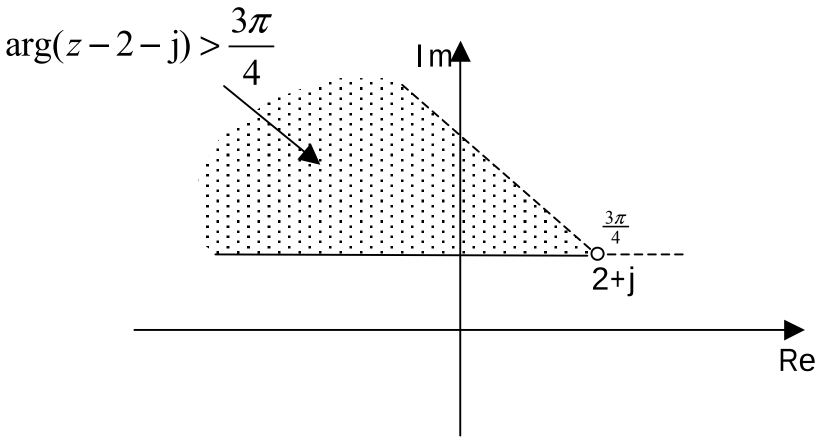
(ii) $\arg(z-2-\mathrm{j}) > \frac{3\pi}{4}$

Solution

(i) $\arg(z+1+2\mathrm{j}) \le \frac{\pi}{3}$ means that $\arg(z+1+2\mathrm{j})$ can take any value between $-\pi$ and $\frac{\pi}{3}$, including $\frac{\pi}{3}$ but not $-\pi$. The half-line from $-1 - 2\mathrm{j}$ in the direction $-\pi$ is therefore shown dotted, and the half-line from $-1 - 2\mathrm{j}$ in the direction $\frac{\pi}{3}$ is shown as solid. The point $-1 - 2\mathrm{j}$ is not included in the region, since $\arg(z+1+2\mathrm{j})$ is not defined where $z = -1 - 2\mathrm{j}$, so this point is shown by an open circle.



(ii) $\arg(z-2-\mathrm{j}) > \frac{3\pi}{4}$ means that $\arg(z-2-\mathrm{j})$ can take any value between $\frac{3\pi}{4}$ and π, including π but not $\frac{3\pi}{4}$ since the inequality involves > rather than ≥. The half-line from 2 + j in the direction $\frac{3\pi}{4}$ is therefore shown dotted, and the half-line from 2 + j in the direction π is shown solid. The point 2 + j is not included in the region, since $\arg(z-2-\mathrm{j})$ is not defined where $z = 2 + \mathrm{j}$, so this point is shown by an open circle.


arg(z − 2 − j) > 3π/4
Im
3π/4
2+j
Re