Core 1

Coordinate Geometry

Section 2: Curves and circles

Notes and Examples

These notes and examples contain subsections on

- **The equation of a circle**
- **Finding the equation of a circle**
- **Circle geometry**
- **The intersection of a line and a curve**
- **The intersection of two curves**

The equation of a circle

Start this section by looking at the **Circles dynamic spreadsheet**. Select the *Circle Equations* sheet. First, set the centre of the circle to be the origin and vary the radius. Look at how the equation of the circle changes.

Now vary the coordinates of the centre of the circle, and look at how the equation of the circle changes.

You can also explore equations of circles using the Flash resources **Equation of a circle centre O** and **Equation of a circle centre (a, b)**.

You should find out the following results, which you need to learn:

The general equation of a circle, centre the origin and radius r is

$$x^2 + y^2 = r^2$$

The general equation of a circle, centre (a, b) and radius r is

$$(x-a)^2 + (y-b)^2 = r^2$$

Make sure you understand why these equations describe circles. See page 62 in the textbook for help.

Example 1
For each of the following circles find (i) the coordinates of the centre and (ii) the radius.
(a) $x^2 + y^2 = 49$
(b) $(x + 2)^2 + (y - 6)^2 = 9$

Solution

(a) $x^2 + y^2 = 49$ can be written as $x^2 + y^2 = 7^2$.

(i) The coordinates of the centre are (0, 0)

(ii) The radius is 7.

This is a particular case of the general form $x^2 + y^2 = r^2$ which has centre (0, 0) and radius r.

(b) $(x + 2)^2 + (y - 6)^2 = 9$ can be written as $(x - (-2))^2 + (y - 6)^2 = 3^2$.

(i) The coordinates of the centre are (–2, 6)

(ii) The radius is 3.

This is a particular case of the general form $(x - a)^2 + (y - b)^2 = r^2$ which has centre (a, b) and radius r.

For practice in examples like the one above, try the interactive questions **Finding the radius and centre of a circle** (*circle equation in its simplest form*).

Sometimes the circle equation needs to be rearranged into its standard form before you can find the centre and radius.

Example 2

Show that the equation $x^2 + y^2 + 4x - 6y - 3 = 0$ represents a circle, and find its centre and radius.

Solution

The general equation of a circle is $(x-a)^2 + (y-b)^2 = r^2$

Multiplying out: $x^2 - 2ax + a^2 + y^2 - 2by + b^2 = r^2$

$$x^2 + y^2 - 2ax - 2by + a^2 + b^2 - r^2 = 0$$

Comparing with the original equation: $-2a = 4 \Rightarrow a = -2$

$$-2b = -6 \Rightarrow b = 3$$

$$a^2 + b^2 - r^2 = -3 \Rightarrow 4 + 9 - r^2 = -3$$

$$\Rightarrow r^2 = 16$$

The equation can be written as $(x+2)^2 + (y-3)^2 = 4^2$

This is the equation of a circle, centre (-2, 3), radius 4.

For practice in examples like the one above, try the interactive questions **Finding the radius and centre of a circle** (*circle equation in its expanded form*).

In the example above, you are using the technique of **completing the square**, which is covered briefly in Chapter 1 (pages 19 – 20), and in more depth in Chapter 3 (pages 98 – 99).

Finding the equation of a circle

In Section 1 you looked at different ways of finding the equation of a line. You can find the equation of a line from the gradient and the intercept, or from the gradient and one point on the line, or from two points on the line.

In the same way, there are several ways of finding the equation of a circle, depending on the information available.

Finding the equation of a circle from the radius and centre

Example 3
Find the equation of each of the following.
(a) a circle, centre (0, 0) and radius 4.
(b) a circle, centre (3, –4) and radius 6.

Solution
(a) The equation of a circle centre the origin is $x^2 + y^2 = r^2$

$r = 4$ so the equation is $x^2 + y^2 = 4^2$
i.e. $x^2 + y^2 = 16$

(b) The equation of a circle centre (a, b) and radius r is $(x - a)^2 + (y - b)^2 = r^2$

$a = 3$, $b = -4$ and $r = 6$ so the equation is $(x - 3)^2 + (y - (-4))^2 = 6^2$
i.e. $(x-3)^2 + (y+4)^2 = 36$

Finding the equation of a circle from its centre and one point on its circumference

If you know the centre of the circle and one point on its circumference, you can find the radius by calculating the distance between these two points. You can then find the equation of the circle.

Example 4
Find the equation of the circle, centre (1, -2), which passes through the point (-2, -3).

Solution
The distance r between (1, -2) and (-2, -3) is given by:

$$r = \sqrt{(1-(-2))^2 + (-2-(-3))^2}$$
$$= \sqrt{3^2 + 1^2}$$
$$= \sqrt{10}$$

The radius of the circle is therefore $\sqrt{10}$.
The equation of the circle is $(x-1)^2 + (y+2)^2 = 10$

For practice in examples like the one above, try the interactive questions **Find the equation of a circle**.

Finding the equation of a circle from three points on its circumference
To find the equation of a line, you need the coordinates of two points on the line. To find the equation of a circle, you need the coordinates of three points on the circumference of the circle.

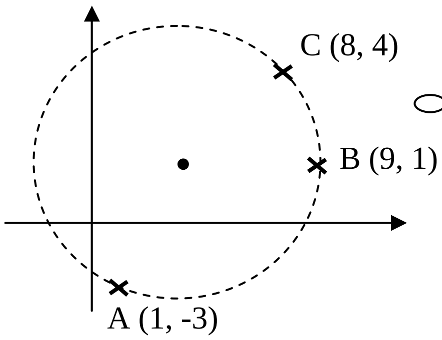
One method is illustrated by the **Circles dynamic spreadsheet**. Select the sheet *Circumcentre* and follow the instructions on the sheet. This demonstration shows that the centre of the circle is the intersection of the perpendicular bisector of each pair of points.

To find the centre of a circle through three points A, B and C, it is sufficient to find two of the perpendicular bisectors. For example, you can find the equations of the perpendicular bisectors of AB and BC, and then solve these equations simultaneously to find the point of intersection, i.e. the centre of the circle.

You can then use the coordinates of the centre and one of the three points A, B and C to find the radius of the circle (as in Example 4), and hence find the equation of the circle.

Example 5
Find the equation of the circle passing through A (1, –3), B (9, 1) and C (8, 4).

Solution



A sketch is often helpful. The sketch does not need to be accurate. It gives some idea of roughly where the centre is, so you can check your answer is reasonable.

You want to find the equation of the perpendicular bisector of AB. This is perpendicular to AB and passes through the midpoint M of AB.

The gradient of AB is found by using $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

$$m = \frac{1-(-3)}{9-1} = \frac{4}{8} = \frac{1}{2}$$

Note: Looking at the sketch we expect the gradient of AB to be positive.

Using $m_1 m_2 = -1$, the gradient of the perpendicular bisector is –2.

The midpoint M of AB is found by using $M = \left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right)$.

You are given A(1, -3) and B(9, 1) so M is $\left(\dfrac{1+9}{2}, \dfrac{-3+1}{2}\right) = (5, -1)$

The perpendicular bisector is found using $y - y_1 = m(x - x_1)$ with $(x_1, y_1) = (5, 1)$ and $m = -2$.

so $y - (-1) = -2(x - 5)$

$y + 1 = -2x + 10$

$y = -2x + 9$ (equation I)

> Next, use the same method to find the perpendicular bisector of BC.

The gradient of BC is $\dfrac{4-1}{8-9} = -3$

> Note: Looking at the sketch, we expect the gradient of BC to be negative.

Therefore the gradient of the perpendicular bisector of BC is $\dfrac{1}{3}$.

The midpoint N of BC is $\left(\dfrac{9+8}{2}, \dfrac{1+4}{2}\right)$ so N is (8.5, 2.5).

The equation of the perpendicular bisector is $y - 2.5 = \frac{1}{3}(x - 8.5)$

$3(y - 2.5) = x - 8.5$

$3y - 7.5 = x - 8.5$

$3y = x + 1$ (equation II)

> Next, find the coordinates of the centre of the circle by solving equations (I) and (II) simultaneously.

$y = -2x + 9$ (equation I)

$3y = x + 1$ (equation II)

Substituting (I) into (II) $3(-2x + 9) = x + 1$

$-6x + 27 = x + 1$

$28 = 7x$

$x = 4$

Substituting $x = 4$ into equation (I) gives $y = -2(4) + 9 = 1$

So the coordinates of the centre are (4, 1).

> Note: Looking at the sketch this appears to be a plausible result.

The radius is the distance between the centre (4, 1) and a point on the circumference such as (9, 1). This can be found by using $d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$.

$$\text{radius} = \sqrt{(9-4)^2 + (1-1)^2} = \sqrt{25} = 5$$

Finally, using the general form $(x - a)^2 + (y - b)^2 = r^2$ with $a = 4$, $b = 1$ and $r = 5$ the equation of the circle is

$$(x - 4)^2 + (y - 1)^2 = 25.$$

Note: You should check that each of the points A, B and C satisfy this equation.

Circle geometry

The three facts about circles given on pages 63 and 64 are important. They often help to solve problems involving circles.

1. The angle in a semicircle is a right angle.
   See the Flash resource **The angle in a semicircle** for a demonstration.

2. The perpendicular from the centre of a circle to a chord bisects the chord.
   See the Flash resource **Perpendicular to a chord** for a demonstration.

3. The tangent to a circle is perpendicular to the radius at that point
   See the Flash resource **Tangent and radius** for a demonstration.

Keep these properties in mind when dealing with problems involving circles.

The intersection of a line and a curve

Just as the point of intersection of two straight lines can be found by solving the equations of the two lines simultaneously, the point(s) of intersection of a line and a curve can be found by solving their equations simultaneously.

In many cases, the equations of both the line and the curve are given as an expression for y in terms of x. When this is the case, a sensible first step is to equate the expressions for y, as this leads to an equation in x only.

Example 6
Find the coordinates of the points where the line $y = x + 2$ meets the curve $y = x^2 - 3x + 5$.

Solution

$x^2 - 3x + 5 = x + 2$

$x^2 - 4x + 3 = 0$

$(x - 3)(x - 1) = 0$

$x = 3$ or $x = 1$

Equate the expressions for y to give an equation in x only

When $x = 3$ then $y = 3 + 2 = 5$
When $x = 1$ then $y = 1 + 2 = 3$.

Substitute the x values into the equation of the line

The points where the line meets the curve are (3, 5) and (1, 3).

You should check that each of these points satisfies the equation of the curve. (You have already used the equation of the line to find the y-values).

Notice that this problem involved solving a quadratic equation, which in this case had two solutions, showing that the line crossed the curve twice. However, the quadratic equation could have had no solutions, which would indicate that the line did not meet the curve at all, or one repeated solution, which would indicate that the line touches the curve.

You can look at some examples with the Flash resource **Intersection of a curve and a line**.

For practice in examples like the one above, try the interactive questions **Quadratic and line intersection**.

The next example looks at the intersection of a line and a circle. Before reading this example, look at the **Circles dynamic spreadsheet**. Select the sheet *Circle and a line*. Try varying the equation of the line and/or the circle, and make sure that you can see that there may be two intersections, no intersections or one intersection (in which case the line touches the circle).

You can also look at the Flash resource **Intersection of a circle and a line**.

Example 7

Find the coordinates of the point(s) where the circle $(x+2)^2 + (y-1)^2 = 9$ meets

(i) the line $y = 5$
(ii) the line $x = 1$
(iii) the line $y = 2 - x$

Solution

(i) Substituting $y = 5$ into the equation of the circle:

$$(x+2)^2 + (5-1)^2 = 9$$
$$(x+2)^2 + 16 = 9$$
$$(x+2)^2 = -7$$

The expression $(x + 2)^2$ cannot be negative

There are no solutions. The line does not meet the circle.

(ii) Substituting $x = 1$ into the equation of the circle:

$$(1+2)^2 + (y-1)^2 = 9$$
$$9 + (y-1)^2 = 9$$
$$(y-1)^2 = 0$$
$$y = 1$$

The point is on the line $x = 1$, so its x-coordinate must be 1.

The line touches the circle at (1, 1).

(iii) Substituting $y = 2 - x$ into the equation of the circle:

$$(x+2)^2 + (2-x-1)^2 = 9$$
$$(x+2)^2 + (1-x)^2 = 9$$
$$x^2 + 4x + 4 + 1 - 2x + x^2 = 9$$
$$2x^2 + 2x - 4 = 0$$
$$x^2 + x - 2 = 0$$
$$(x-1)(x+2) = 0$$
$$x = 1 \text{ or } x = -2$$

Substitute the x values into the equation of the line to find the y-coordinates.

When $x = 1$, $y = 2 - 1 = 1$

When $x = -2$, $y = 2 - (-2) = 4$

The line crosses the circle at (1, 1) and (–2, 4).

For practice in examples like the one above, try the interactive questions **Circle and line intersection**.

The intersection of two curves

As before, the intersections of two curves can be found by solving the equations of the curves simultaneously. As in Example 4, in many cases a sensible first step is to equate the expressions for y.

Example 8

Find the coordinates of the points where the curve $y = x^2 - 6x + 5$ intersects the curve $y = -2x^2 + 12x - 19$.

Equate the expressions for y to give an equation in x only

Solution

$x^2 - 6x + 5 = -2x^2 + 12x - 19$

$\Rightarrow 3x^2 - 18x + 24 = 0$

$\Rightarrow x^2 - 6x + 8 = 0$

$\Rightarrow (x - 2)(x - 4) = 0$

$\Rightarrow x = 2$ or $x = 4$

There is a common factor of 3, so divide by 3 before factorising.

When $x = 2$, $y = (2)^2 - 6(2) + 5 = -3$
When $x = 4$, $y = (4)^2 - 6(4) + 5 = -3$

Substitute the x values into the equation of one of the curves (in this case the first one).

The points of intersection are $(2, -3)$ and $(4, -3)$.

You should check that each of these points satisfy the equation of the second curve. (You have already used the equation of the first curve to find the y-values).